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GENERAL SOLUTIONS OF OPTIMUM PROBLEMS IN NONSTATIONARY FLIGHT

By Angelo Miele

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GENERAL SOLUTIONS OF OPTIMUM PROBLEMS IN NONSTATIONARY FLIGHT*[1]

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SUMMARY

A general method concerning optimum problems in nonstationary flight is developed and discussed.

Best flight techniques are determined for the following conditions: climb with minimum time, climb with minimum fuel consumption, steepest climb, descending and gliding flight with maximum time or with maximum distance.

Optimum distributions of speed with altitude are derived assuming constant airplane weight and neglecting curvatures and squares of path inclination in the projection of the equation of motion on the normal to the flight path.

The results of this paper differ from the well-known results obtained by neglecting accelerations with one exception, namely, the case of gliding with maximum range.

The paper is concluded with criticisms and remarks concerning the physical nature of the solutions and their usefulness for practical applications.

SYMBOLS

C	weight of fuel consumed
$F_a = V_z/V_{zu} = \sin\theta/\sin\theta_u$	acceleration factor: ratio of effective rate of climb to the one computed neglecting accelerations

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g acceleration of gravity

P lift

q weight of fuel consumed per unit time

Q airplane weight

R aerodynamic drag

s distance flown

s_0 horizontal projection of the distance flown

t time

T thrust

V speed

$V_z = V \sin \theta$ effective rate of climb

$V_{zu} = V \sin \theta_u$ rate of climb computed from the equations of uniform flight

Z altitude

Z_* altitude of tropopause

θ effective path inclination (positive for climbing flight)

θ_u path inclination computed from the equation of uniform flight (positive for climbing flight)

1. INTRODUCTION

In the past airplane performances have usually been determined by neglecting accelerations, which results in great simplifications in the equations of motion.

The small curvatures associated with the usual paths of an aircraft in a vertical plane justify the assumption of zero centrifugal forces. On the other hand, the inertia tangential forces (often logically disregarded in the study of performances of many conventional aircraft) must be taken into account in the analysis of high-performance jet airplanes, because of the large values of both the velocity and its variation with the altitude.

For instance, it is well known that the speed for best climb of any type of aircraft increases with altitude (incompressible flow). This means that the total energy developed by the power plant is not only used to work against the aerodynamic drag and the earth's gravitational field, but also to increase the kinetic energy of the aircraft.

When the terms due to acceleration are neglected, then the estimated climbing performances are too optimistic; the values of time and fuel consumption calculated in this way are less than the actual ones.

On the other hand it appears that compressibility effects can sometimes decrease the speed for best climb as altitude increases, especially in the stratosphere and for aircraft with high wing loading; in this case the rate of climb computed from the equation of uniform flight is less than the actual one.

The above-mentioned reasons emphasize the need for an analysis of optimum flight conditions based on the consideration of the nonuniform character of the motion.

The first attack to the problem was performed by F. C. Phillips (ref. 1), who proposed a kinetic energy correction to the results predicted with the equations of a uniform flight. The calculation of such a correction was carried out by assuming a distribution of speed with altitude identical with the one which maximizes the rate of climb computed without accelerations. (See ref. 4, also.)

Other studies, due to Otten (ref. 2) and Hayes (ref. 3), extended Phillips' results by taking into consideration the true path inclination and the effects of compressibility.

But only in recent years the problem has been considered and analyzed in its entirety. In particular, the author (ref. 5) has solved the problem of climb with minimum time by using a transformation based on Green's theorem, while Lush (ref. 6), following a study due to Kaiser (ref. 7), has treated the same problem by an elegant graphic-analytic method based on the concept of energy height ($Z_e = Z + V^2/2g$).

The present paper generalizes a method described in a preceding study (ref. 5) and extends its results to the following types of flight: climb with minimum time; climb with minimum fuel consumption; steepest climb; descending or gliding flight with maximum time or with maximum horizontal distance.

For each case the optimum technique of flight is determined; that is to say the function $V = V(Z)$ which, from given initial conditions (V_1, Z_1) to fixed final conditions (V_2, Z_2), will maximize or minimize the time, the fuel consumption or the distance.

Results are discussed and criticized. Their limitations are pointed out; their field of applicability is indicated for the different types of today's engine groups (reciprocating engine, air-breathing jet engines, rockets).

2. BASIC HYPOTHESES

The following hypotheses are basic for all the work:

(a) Airplane weight is assumed constant.

(b) Curvatures and squares of path inclination are assumed negligible with regard to their effects on that part of the drag depending on the angle of attack.

(c) Power plant is of an unspecified type; but its thrust and rate of fuel consumption are assumed to be functions of the following nature:

$$T = T(V, Z) \tag{1}$$

$$q = q(V, Z) \tag{2}$$

(d) Angle between the vectors T, V is not taken into consideration.

(e) Only flight paths restricted to a vertical plane are considered.

(f) The aerodynamic lag is disregarded; the air forces are calculated as in steady flight.

3. FUNDAMENTAL EQUATIONS

The following scalar expressions can be derived projecting the fundamental equation of the motion on the tangent and on the normal to the flight path:

$$T - R - Q\left[1 + \frac{V}{g}\frac{dV}{dZ}\right]\sin\theta = 0 \tag{3}$$

$$P - Q\left[\cos\theta + \frac{V^2}{g}\frac{d\theta}{dZ}\sin\theta\right] = 0 \tag{4}$$

According to the preceding hypothesis (b), equation (4) can be substituted by

$$P - Q = 0 \tag{5}$$

This approximation is important. As a matter of fact, the rate of climb given by

$$V_z = V \sin \theta = \frac{(T - R)V/Q}{1 + \frac{V}{g}\frac{dV}{dZ}} \tag{6}$$

becomes a function of V, Z, and dV/dZ only, as can be seen from equations (1), (5), and (6), the expressions for the aerodynamic forces and the polar.

Equation (6) shows that the effective rate of climb and the sine of the effective path inclination can be expressed as the product of the corresponding values obtained for uniform flight

$$V_{zu} = (T - R)V/Q \tag{7}$$

$$\sin \theta_u = (T - R)/Q \tag{8}$$

by the correction factor

$$F_a = \frac{1}{1 + \frac{1}{2g}\frac{dV^2}{dZ}} \tag{9}$$

which expresses the effects associated with the nonuniform character of the motion.

It should be noted that the preceding equations are general; therefore, they contain those corresponding to gliding flight as a special case $(T = 0)$.

4. CLIMBING FLIGHT

The following cases of flight are discussed:

(a) Climb with minimum time.

(b) Climb with minimum fuel consumption.

(c) Steepest climb.

It is assumed that the thrust at all times is greater than the aerodynamic drag.

4-a. Climb With Minimum Time

The time necessary to fly from given initial conditions (V_1, Z_1) to fixed final conditions (V_2, Z_2) is

$$t = \int_1^2 \frac{dZ}{V_z} = \int_1^2 (\Phi \, dV + \Psi \, dZ) \tag{10}$$

where

$$\Phi = \frac{Q}{g(T - R)} = \phi(V, Z) \tag{11}$$

$$\Psi = \frac{Q}{V(T - R)} = \psi(V, Z) \tag{12}$$

Here it is desired to determine the best flight technique; that is to say, the particular speed-height relationship $V = \bar{V}(Z)$ which minimizes integral (10).

The investigation is simplified if the properties of the function

$$\omega(V, Z) = \frac{\partial \Psi}{\partial V} - \frac{\partial \Phi}{\partial Z} \tag{13}$$

are used instead of the application of variational methods[2].

The curve $\omega = 0$ divides that zone of the (V, Z) plane which is of practical interest for flight operations (fig. 1) into two regions: A, where $\omega < 0$; B where $\omega > 0$.

Four cases of flight (i.e. four types of boundary conditions) are possible according to the relative positions of points 1 and 2 with respect to the curve $\omega = 0$:

Case I: Point 1 in zone A; point 2 in zone B.

Case II: Point 1 in zone A; point 2 in zone A.

Case III: Point 1 in zone B; point 2 in zone A.

Case IV: Point 1 in zone B; point 2 in zone B.

Here, only the first case is analyzed with the restriction that $Z_1 \leq Z \leq Z_2$.

The optimum flight technique is the following:

(1) Acceleration at constant altitude Z_1 from V_1 to the speed (V_M) defined by $\omega(V, Z_1) = 0$.

(2) Climb from Z_1 to Z_2 using the distribution of velocities defined by $\omega(V, Z) = 0$.

(3) Acceleration at constant altitude Z_2 from the speed (V_N) defined by $\omega(V, Z_2) = 0$ to V_2.

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[2]The exact study of the problem, made using equation (4) instead of equation (5) enables one to take into account four boundary conditions, e.g. values of V, θ corresponding to initial and final altitudes.

However, the approximations involved in the present analysis permit one to impose only two boundary conditions, e.g., values of V at the initial and final altitudes. In fact, values of θ are a consequence, because of equations (3) and (5), of the same solution which is being investigated in this paper.

The minimal nature of the aforementioned speed-height relationship will be proved by showing that the following inequality is satisfied:

$$\Delta t = t - \tau = \oint_{1K2NKM1} (\Phi \, dV + \Psi \, dZ) > 0 \tag{14}$$

where τ is the time necessary to pass from 1 to 2 using the optimum path 1MN2 and t is the time necessary to fly along the arbitrary path 1K2 (which, however, shall be physically possible under the imposed condition $T > R$).

The line integral (14) can be separated into two integrals associated with the closed circuits K2NK and KM1K

$$\Delta t = \oint_{K2NK} (\Phi \, dV + \Psi \, dZ) + \oint_{KM1K} (\Phi \, dV + \Psi \, dZ) \tag{15}$$

By Green's theorem the line integrals contained in equation (15) can be transformed into surface integrals connected with the areas S_A and S_B encompassed by the above-mentioned boundaries

$$\Delta t = \iint_{S_B} \left(\frac{\partial \Psi}{\partial V} - \frac{\partial \Phi}{\partial Z}\right) dV \, dZ - \iint_{S_A} \left(\frac{\partial \Psi}{\partial V} - \frac{\partial \Phi}{\partial Z}\right) dV \, dZ \tag{16}$$

Because ω is positive in S_B and negative in S_A, it follows that $\Delta t > 0$. Thus the theorem is proved[3]. It must be emphasized that the optimum path includes a line MKN along which the distribution of speed is defined by

$$\frac{\partial\Psi}{\partial V} - \frac{\partial\Phi}{\partial Z} = 0 \tag{17}$$

or according to equations (7), (11), and (12) by

$$\frac{\partial(T - R)V}{\partial V} = \frac{V}{g}\frac{\partial(T - R)V}{\partial Z} \tag{18}$$

The same problem can be studied with the help of the Calculus of Variations. But, probably because of the basic hypotheses, only the central part MKN of the optimum path can be determined in that way[4].

4-b. Climb With Minimum Fuel Consumption

The weight of fuel necessary to fly from (Z_1, V_1) to (Z_2, V_2) is

$$C = \int_1^2 \frac{q}{V_z}\, dZ = \int_1^2 (\Phi_1\, dV + \Psi_1\, dZ \tag{19}$$

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[3]For case II which seems to have some practical interest a quasi-optimum solution (when V_2 is not much less than V_N) could be the following:

(1) Acceleration at constant altitude Z_1 from V_1 to V_M.

(2) Climb using the speed distribution defined by $\omega = 0$ until the altitude Z_3 corresponding to V_2 is reached.

(3) Climb at constant velocity V_2 from Z_3 to Z_2.

[4]The study of the problem of absolute minimum without the restrictive condition $Z_1 \leq Z \leq Z_2$ leads to an optimum trajectory composed of:

(1) A central pattern along which the distribution of velocities is defined by equation (17).

(2) Two initial and final branches that must be flown in vertical flight (ascending or descending) according to the boundary conditions of the problem (see appendix).

where

$$\Phi_1 = q\Phi = \frac{qQ}{g(T - R)} \tag{20}$$

$$\Psi_1 = q\Psi = \frac{qQ}{V(T - R)} \tag{21}$$

The problem of finding the special function $V = V(Z)$ which minimizes integral (19) is analogous to the preceding one; hence, the solution is of the same type. With reference to case I, and again with the restriction that $Z_1 \leq Z \leq Z_2$ the best flight technique comprises two accelerated motions at the initial and final altitudes Z_1, Z_2 and a central climbing path along which the distribution of speeds is defined by

$$\omega_1(V, Z) = \frac{\partial \Psi_1}{\partial V} - \frac{\partial \Phi_1}{\partial Z} = 0 \tag{22}$$

or, according to equations (20) and (21), by

$$\frac{\partial}{\partial V}\left[\frac{(T - R)V}{q}\right] = \frac{V}{g}\frac{\partial}{\partial Z}\left[\frac{(T - R)V}{q}\right] \tag{23}$$

4-c. Steepest Climb

The total distance traveled by the aircraft flying from (Z_1, V_1) to (Z_2, V_2) is

$$s = \int_1^2 \frac{dZ}{\sin \theta} = \int_1^2 (\Phi_2 \, dV + \Psi_2 \, dZ) \tag{24}$$

where

$$\Phi_2 = V\Phi = \frac{VQ}{g(T - R)} \tag{25}$$

$$\Psi_2 = V\Psi = \frac{Q}{(T - R)} \tag{26}$$

The speed-height function $V = V(Z)$ which minimizes integral (24) is analogous to those minimizing integrals (10) and (19).

With reference to case I and to the central pattern flown at varying altitude the best distribution of speeds is defined by

$$\omega_2(V, Z) = \frac{\partial\Psi_2}{\partial V} - \frac{\partial\Phi_2}{\partial Z} = 0 \tag{27}$$

or, according to equations (25) and (26), by

$$\frac{\partial(T - R)}{\partial V} = \frac{V}{g}\frac{\partial(T - R)}{\partial Z} \tag{28}$$

The horizontal projection of the distance traveled is

$$s_0 = \int_1 \frac{dZ}{\tan\theta} \tag{29}$$

If the path inclination is sufficiently small, so that it is justified to assume $\sin\theta \cong \tan\theta$ equation (29) becomes identical with equation (24). It follows that the distribution of speeds defined by equation (28) minimizes the horizontal projection of the distance traveled and is therefore the best from the point of view of the so-called "steepest climb".

5. DESCENDING FLIGHT

The following conditions of flight are examined:

(a) Descending flight with maximum time.

(b) Descending flight wih maximum horizontal distance.

Thrust is assumed at all times to be less than the aerodynamic drag.

5-a. Descending Flight With Maximum Time

Now, case III is examined (the same nomenclature of the preceding paragraph is used); namely, deceleration from high altitude and high speed to low altitude and low speed. This condition of flight is of more practical interest than cases I, II, and IV.

Under the restrictive condition $Z_1 \geq Z \geq Z_2$, the best flight technique is the following (see fig. 2):

(1) Deceleration at constant altitude Z_1 from V_1 to V_M defined by $\omega(V, Z_1) = 0$ (the function $\omega(V, Z_1)$ is defined by equation (13)).

(2) Descending flight from Z_1 to Z_2 using the distribution of velocities defined by $\omega(V, Z) = 0$.

(3) Deceleration at constant altitude Z_2 from the speed V_N defined by $\omega(V, Z_2) = 0$ to V_2.

This statement can be easily proved using Green's theorem as in the preceding paragraph. It should be noted that the distribution of speeds necessary to climb with minimum time is of the same form as the speed-height relationship required to descend with maximum time. However both distributions are not numerically identical since different thrusts are required.

The results valid for gliding flight may be derived from the above as a limiting process by letting $T \longrightarrow 0$.

The equation for the optimum speed-height function for the central pattern MN flown at variable descending altitude is given by

$$\frac{\partial(RV)}{\partial V} = \frac{V}{g} \frac{\partial(RV)}{\partial Z} \tag{30}$$

5-b. Descending Flight With Maximum Horizontal Distance

The best technique for this type of flight is analogous to that described in the preceding paragraph.

It consists of two decelerations at constant altitude Z_1, Z_2 and of a central pattern defined by $\omega_2(V, Z) = 0$. (Case III).

For gliding flight the best distribution of velocities can be obtained as a particular case of equation (28) for $T \longrightarrow 0$

$$\frac{\partial R}{\partial V} = \frac{V}{g}\frac{\partial R}{\partial Z}. \tag{31}$$

It should be noted that the solution defined by equation (31) is identical with the one that can be obtained from an analysis based on the equations of uniform flight if the variation of the drag with both the Reynolds and Mach numbers is neglected.

As a matter of fact the aerodynamic drag depends, in this latter case, on the dynamic pressure only. Thus the practical solution of equation (31) is given by the equivalent expressions

$$\frac{\partial R}{\partial V} = \frac{\partial R}{\partial Z} = 0 \tag{32}$$

6. REMARKS AND CRITICISMS ON THE ACHIEVED SOLUTIONS

A short review of the obtained results will clarify their physical nature and will be helpful from the point of view of practical applications.

6-a. Comparison Between Stationary and Instationary Solutions[5]

Solutions commonly used in the practical applications of the Mechanics of Flight are those derived from a "stationary" analysis[5].

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[5]Within the limits of the present investigation the term "stationary (instationary) solution" means a solution obtained neglecting (taking into account) inertia tangential forces. This terminology is used for the sake of brevity. The so-called aerodynamic lag is disregarded. In other words the air forces are calculated as in a steady flight.

They are the following:

For climb with minimum time

$$\frac{\partial(TV - RV)}{\partial V} = 0 \tag{33}$$

For climb with minimum fuel consumption

$$\frac{\partial}{\partial V}\left[\frac{TV - RV}{q}\right] = 0 \tag{34}$$

For steepest climb

$$\frac{\partial(T - R)}{\partial V} = 0 \tag{35}$$

For maximum endurance in gliding

$$\frac{\partial(RV)}{\partial V} = 0 \tag{36}$$

For maximum range in gliding

$$\frac{\partial R}{\partial V} = 0 \tag{37}$$

These solutions could also be achieved as a particular case of those given in this paper if one supposes that the motion takes place in an ideal ambient of constant air density. In fact in this latter case the derivatives of T, q, and R with respect to Z vanish and equations (18), (23), (28), (30), and (31) are reduced to equations (33), (34), (35), (36), and (37), respectively.

It is evident that the second members of equations (18), (23), (28), (30), and (31) express synthetically the contribution given by the acceleration to the equation defining the optimum speeds.

6-b. The Case $\frac{\partial q}{\partial V} = 0$

If power plant is of such a type as to justify the above-mentioned assumption, the "stationary" solutions for minimum time and for minimum fuel consumption become identical.

Therefore, it is logical to suppose that in this case the "instationary" solution for minimum time will not differ too much from the one optimum for minimum fuel consumption. Some numerical calculations have confirmed this last concept.

6-c. The Case q = Constant

The "instationary" solutions for minimum time and for minimum fuel consumption become identical (rocket-powered aircraft).

6-d. Discontinuity of the Solutions at the Tropopause

The optimum speed-height relationships given by equations (18), (23), (28), (30), and (31) have a discontinuity at the tropopause. This fact is related to our manner of conceiving the standard atmosphere in which the derivatives with respect to h of the density, temperature and pressure have two values at $Z = Z_*$.

As a consequence, for any case of flight there are two optimum speeds at the tropopause, the one being deduced by introducing into equations (18), (23), (28), (30), and (31) the properties of the standard troposphere (V_t) and the other by introducing into the same equations the properties of the standard stratosphere (V_s).

The mechanical meaning of the aforementioned discontinuity[6] may be understood using Green's theorem as in the previous sections. For instance, in the case of a turbojet aircraft this indicates the necessity of accelerating the aircraft at $Z = Z_*$ from $V_t(V_s)$ to $V_s(V_t)$ if the

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[6]Analogously a discontinuity in the optimum speed-height relationship can be detected at the critical altitude (Z_c) of an aircraft powered by a reciprocating engine-propeller combination. This fact depends on the existence of two values of $\delta P/\delta Z$ at $Z = Z_c$ (P = shaft horsepower of a conventional engine).

achievement of the best climbing performances is desired (see fig. 3). As a consequence the optimum flight technique for climb with minimum time (case I) from $(V_1, Z_1 < Z_*)$ to $(V_2, Z_2 > Z_*)$ under the limiting conditions $Z_1 \leq Z \leq Z_2$, consists of:

(1) Acceleration at constant altitude Z_1 from V_1 to V_M defined by $\omega(V, Z_1) = 0$.

(2) Climb from (V_M, Z_1) to (V_t, Z_*) using the speed-height relationship defined by $\omega(V, Z) = 0$.

(3) Acceleration at constant altitude Z_* from V_t to V_s.

(4) Climb from (V_s, Z_*) to (V_N, Z_2) using again the distribution of speeds defined by $\omega(V, Z) = 0$.

(5) Acceleration at constant altitude Z_2 from V_N to V_2.

6-e. Hypothesis Concerning Curvature and Path Inclination

The practical consequence of the hypothesis (b) of section (2) is an approximate calculation of that part of the drag which depends on the lift. The errors involved have small importance for many of the cases of flight here considered.

In any case the following concept should be emphasized: the use of the solutions here achieved is logical only if the errors associated with the neglect of curvatures and squares of the path inclination are small with respect to those avoided taking into account the tangential accelerations.

A systematic investigation of the exact limits of applicability of the present theory to the various types of modern aircraft is beyond the scope of this report. However, it seems possible to anticipate that the hypotheses concerning the curvatures and the path inclinations are justified in the following cases:

(1) Climb with minimum time and with minimum fuel consumption: jet-propelled aircraft and conventional aircraft with high wing-and-power loadings.

(2) Steepest climb: turbojet aircraft with low specific thrust $(\overline{T/Q})$ and good aerodynamic efficiency (results concerning the steepest climb are not only influenced by the approximations made in the projection of the equation of motion on the normal to the flight path but also by the substitution $\tan\theta$ in lieu of $\sin\theta$ in the projection of the equation of motion on the tangent to the flight path).

(3) Gliding flight: airplanes having high wing loading and good aerodynamic efficiency.

6-f. Hypothesis Concerning the Weight of the Aircraft

The weight of the aircraft changes during the flight because of the fuel consumption. Consequently, the true optimum speed-height relationships are somewhat different from the ones previously derived. The true rates of climb are greater than those calculated by assuming Q = Constant. According to the practical values of dQ/dt the following remarks are formulated:

(1) Aircraft powered by air-breathing engines: The speed-height functions previously derived are substantially correct. If more precision is desired the weight changes can approximately be taken into account by iterating the calculations as follows:

(a) Calculate the optimum distributions of speeds with equations (18), (23), and (28) according to the case of flight and supposing Q = Constant.

(b) Determine the approximate values of the fuel consumption on the basis of the above-mentioned distributions of speeds.

(c) Calculate the instantaneous weights of the aircraft at any altitude.

(d) Determine the new optimum speed-height functions by introducing into equations (18), (23), and (28) the instantaneous weights. Calculate also the new values of integrals (10), (19), and (24).

(2) Rocket-powered aircraft: From a purely theoretical standpoint the results here derived cannot be considered valid for this kind of aircraft because of the important dynamical effects associated with the changes of the airplane weight.

Notwithstanding, the author believes that the obtained solutions are very close to the true solutions for tropospheric flight above all if iterative procedures, like the one outlined above, are applied.

That depends on the fact that the only term of equation (18) depending on the weight is that part of the drag which is a function of the lift; that is to say the so-called "induced" drag, which is small at low altitudes because of the low angles of attack used by rocket-powered aircraft in the climbing flight.

6-g. Additional Remarks Concerning Centripetal Accelerations

The neglect of curvatures (and therefore of the deviation times necessary to pass from one branch to another of the optimum path) has a qualitative influence on the results leading to discontinuous solutions; while it is logical to think that the exact study of the problem made with variational methods and using equation (4) instead of equation (5) would bring continuous speed distributions.

Consequently, the results contained in this paper are to be considered as limiting results whose degree of agreement with experiments increase as the ratio of sum of deviation times to the total time decrease.

6-h. Considerations Related to the Method Used in This Paper

The main effect of the hypotheses concerning weight, curvatures and path inclinations has been the possibility of expressing the aerodynamic drag as a function of the type

$$R = R(V, Z) \tag{38}$$

It follows that if the basic hypotheses are changed formulas (18), (23), (28), (30), and (31) may retain their validity provided the drag remain still a function of only the speed and the altitude.

7. CONCLUSIONS

A general method concerning optimum problems in nonstationary flight is developed and discussed. Various conditions of flight in a vertical plane (climb with minimum time, climb with minimum fuel consumption, steepest climb, descending and gliding flight with maximum time or space) are studied; the corresponding best techniques of flight, i.e. the optimum speed-height relationships, are determined.

Each optimum path consists of an initial and a final branch which depend on the boundary conditions of the problem and a central portion which is flown at variable altitude and speed.

Along this central pattern the speed-height relationship obeys the following rule:

If $X = X(V, Z)$ is the function whose maximum or minimum with respect to the speed

$$\frac{\partial X}{\partial V} = 0 \tag{39}$$

defines the best speed-height relationship when a given optimum problem is studied with the equations of uniform flight, then the modified solution of the same problem when the effects of acceleration are considered is defined by

$$\frac{\partial X}{\partial V} = \frac{V}{g} \frac{\partial X}{\partial Z} \tag{40}$$

The optimum speed-height relationships have a discontinuity at the tropopause and differ in general from the solutions based on the assumption of uniform flight with one exception, namely, gliding with maximum range. This latter result is valid if the variation of the drag with both the Mach and Reynolds numbers is neglected.

The practical application of the method given in this paper to the particular case of turbojet aircraft will be published soon.

APPENDIX

PROBLEMS OF ABSOLUTE OPTIMUM

A number of minimal problems have been treated in the preceding paragraphs with the help of some restrictive conditions. For example, the restrictive condition $Z_1 \leq Z \leq Z_2$ has been used to study the climb with minimum time (case I). In other words the problem of accelerating and climbing from low speed and low altitude to high speed and high altitude has been analyzed by considering only paths internal to the region of space limited by the horizontal planes corresponding to the initial and final heights.

However, it may be noted that the method given in this paper may be easily extended to the study of problems of absolute optimum.

If no restrictive condition is imposed to the altitude, the speed-height function optimum to climb with minimum time consists of a central branch whose equation is still $\omega = 0$ and of two initial and final portions which must be flown in vertical flight (ascending or descending) according to the boundary conditions of the problem (see table I and fig. 4).

This statement may be easily proved by applying Green's theorem as shown in section 4.

The main comments concerning the paths indicated in figure 4 are the following:

(a) For jet-propelled aircraft the hypothesis $P = Q$ leads to errors which are small along the line whose equation is $\omega = 0$ and also for the vertical branches 1M and N2 provided they are flown at high speed.

On the other hand the errors may probably be of some importance for the vertical branches if a part of them is to be flown at relatively low speed (that depends evidently on the boundary conditions of the problem).

(b) The optimum speed-height relationships shown in figure 4 have a discontinuous character. On the other hand it is logical to presume that, should the problem be studied with the use of the exact equations of the motion, results would have a continuous character.

Consequently, the results contained in this paper must be considered as limiting results as shown in section 6-g.

In addition, it might be well to bear in mind that, even if an exact study of the problem were possible from a mathematical standpoint, the conclusions would still have to be submitted to other limitations, namely, those imposed by the physiological strength of the pilot and those imposed by the structural strength of the aircraft.

Translated by A. Miele[7]

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TABLE I

SPEED-HEIGHT RELATIONSHIPS OPTIMUM FOR CLIMBING WITH MINIMUM TIME

Case of Flight	Point 1 is in zone	Point 2 is in zone	Speed-height function		
			Branch 1M	Branch MN	Branch N_2
I	A	B	Vertical dive	$\omega = 0$	Vertical dive
II	A	A	Vertical dive	$\omega = 0$	Vertical climb
III	B	A	Vertical climb	$\omega = 0$	Vertical climb
IV	B	B	Vertical climb	$\omega = 0$	Vertical dive

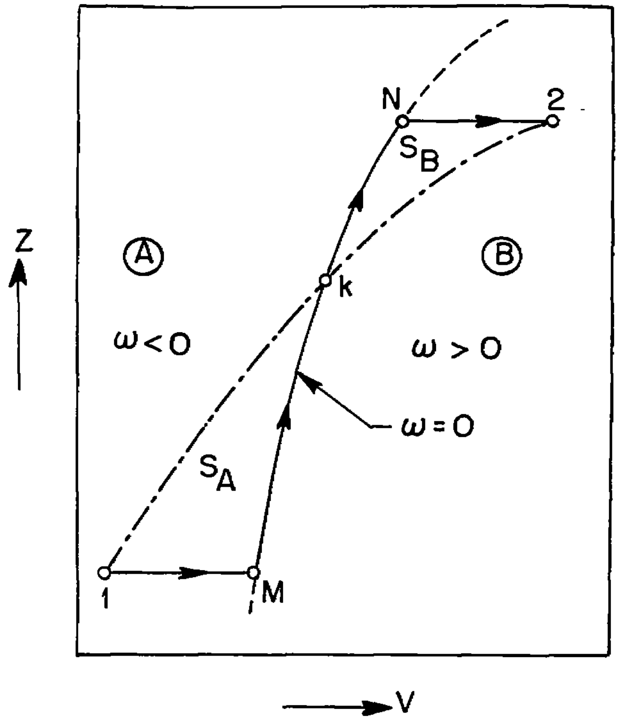

Figure 1.- Speed-height relationship 1MN2 for climb with minimum time; case I; $Z_1 \leq Z \leq Z_2$; initial and final altitudes are either both tropospheric or both stratospheric.

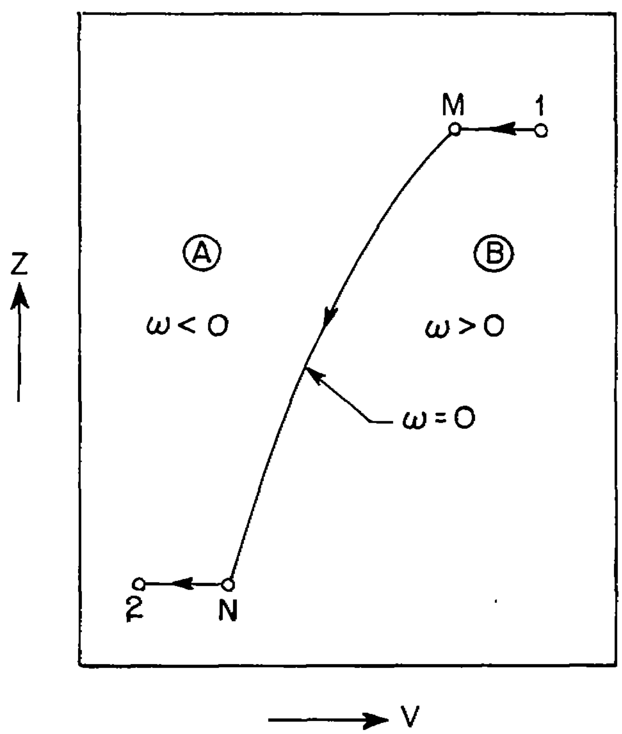

Figure 2.- Speed-height relationship IMN2 for descending flight with maximum time; case III; $Z_1 \geq Z \geq Z_2$; initial and final altitudes are assumed either both tropospheric or both stratospheric.

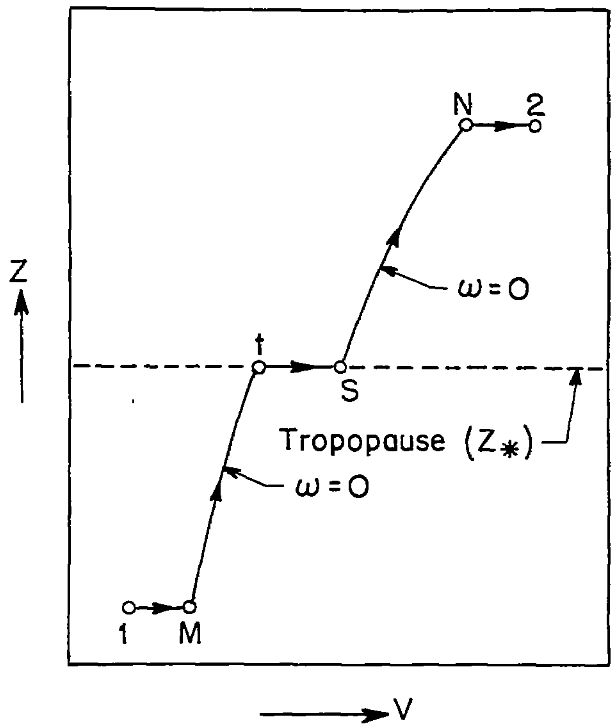

Figure 3.- Speed-height relationship 1MtSN2 for climb with minimum time; case I; $Z_1 \leq Z \leq Z_2$; initial altitude is assumed to be tropospheric; final altitude is assumed to be stratospheric.

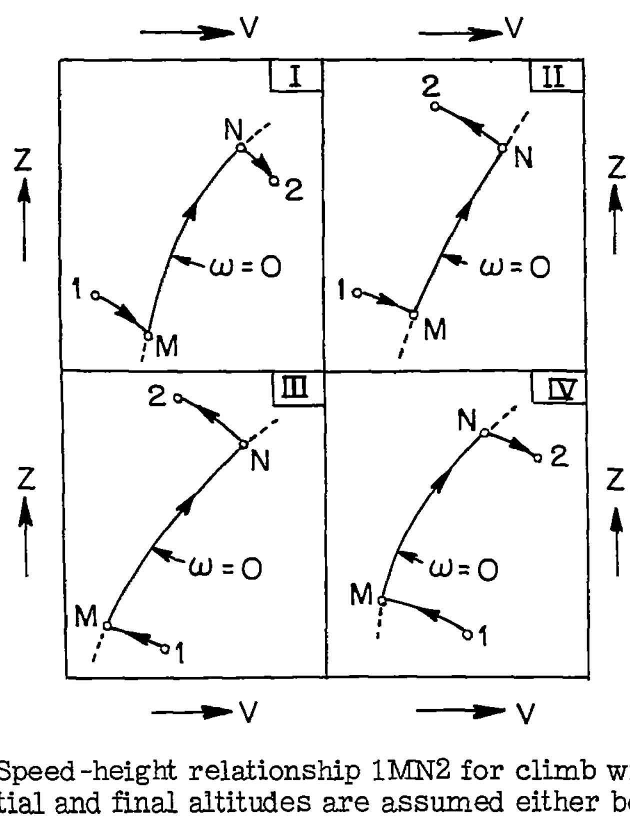

Figure 4.- Speed-height relationship 1MN2 for climb with minimum time; initial and final altitudes are assumed either both tropospheric or both stratospheric.

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